identically, and relation (4.2) takes the form

$$I_1(r_2-r_3)+I_2(r_3-r_1)+I_3(r_1-r_2)=0 \qquad (4.5)$$

The above relation certainly holds in the first, as well as in the second case of integrability, though the conditions for a complementary integral to exist in these cases ($r_1=r_2=r_3$ or $I_1=I_2, r_1=r_2$) are not necessary for the relation to hold. We also note that relation (4.5) does not hold for a homogeneous ellipsoidal body.

5. In the case of an arbitrarily small perturbation in the integrable problem of the motion of a sphere whose centre of mass coincides with its geometrical centre, the necessary conditions for a complementary first integral to exist, analytic in the phase variables, in the class of bodies with ellipsoidal surfaces (a_k and r_k in (2.1) are such, that $a_1^2+a_2^2+a_3^2\neq 0, r_1^2+r_2^2+r_3^2\neq 0$), are combinations of the corresponding conditions of Sect.3 and 4. This proves the following

Theorem. The following three conditions are simultaneously necessary for a complementary first integral to exist, analytic in the phase variables, in the problem of the motion of a heavy rigid ellipsoidal, nearly spherical body, whose centre of mass coincides with its geometrical centre and the moments of inertia are all different: 1) the centre of mass of the ellipsoid coincides with its geometrical centre; 2) the principal axes of the inertia ellipsoid and surface ellipsoid coincide; 3) the moments of inertia of the ellipsoid and the semi-axes of its surface are connected by the relation

$$I_1(\rho_2-\rho_3)+I_2(\rho_3-\rho_1)+I_3(\rho_1-\rho_2)=0$$

The problem of the existence of a complementary analytic integral in the problem of the motion of a body of arbitrary, nearly spherical shape, whose centre of mass coincides with its geometrical centre, is more interesting and more complex. In this case, the first approximation in terms of a small parameter already yields a potential which may represent, generally speaking, an arbitrary function of the direction cosines $\gamma_1, \gamma_2, \gamma_3$, unlike the function H_1 (2.2) representing the sum of the linear and quadratic forms of the variables $\gamma_1, \gamma_2, \gamma_3$.

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PERTURBED MOTION OF A KOVALEVSKAYA TOP*

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Perturbation theory based on the application of Lie series, is used to study a special case of the motion of a rigid body about a fixed point. The equations written in action-angle variables are used in Hamiltonian form. The solutions are obtained in the form of trigonometric series with constant coefficients.

It is assumed that the distribution of mass in the body is clsoe to the distribution in the Kovalevskaya case and the centre of gravity of the body is situated fairly near to the fixed point. The canonical Deprit variables /1/ are used. The motion of the body can be described in these variables by the following set of equations:

$$\frac{d(L,G,H)}{dt}=\frac{\partial F}{\partial(l,g,h)},\quad \frac{d(l,g,h)}{dt}=-\frac{\partial F}{\partial(L,G,H)}$$

Using the condition that the centre of gravity of the body is situated fairly close to the fixed point and the principal moments of inertia A and B differ from each other, we can

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write the Hamiltonian of these equations, in accordance with the order of magnitudies, in the form /2/

$$F = F_0 + F_1$$
$$F_0 = \frac{G^2 - L^2}{2A} - \frac{L^2}{2C} \tag{1}$$
$$F_1 = \frac{A-B}{2AB}(G^2 - L^2)\cos^2 l + \frac{px_c}{G^2}(L\sqrt{G^2 - H^2}\sin l \cos g + G\sqrt{G^2 - H^2}\cos l \sin g + H\sqrt{G^2 - L^2}\sin l)$$

Here A, B, C are the principal moments of inertia for the fixed point, P is the weight of the body and x is the coordinate of the centre of mass in the principal axes of inertia.

We take $\varepsilon = \max\{A - B, x_c\}$ as the small parameter. To solve the equations with the Hamiltonian (1) it is convenient to apply the method of the theory of canonical transformations using Lie series, developed by Hori /3/.

We eliminate the variable l from the Hamiltonian by carrying out the canonical variable transformation

$$L, G, H, l, g, h \to L', G', H', l', g', h'$$

with help of the generating function $S'(L', G', H', l', g', h') = S_1' + S_2' + \ldots$. We assume that the Hamiltonian of the new equations can also be written according to the order of magnitudes in the form $F' = F_0' + F_1' + \ldots$.

According to the Hori method the components of the generating function and the Hamiltonian are obtained from the formulas

$$F_0' = F_0,\ F_1' = F_{1s},\ S_1' = \int F_{1p}dt',\ F_2' = F_{2s} + {}^1/_2\{F_1 + F_1', S_1'\}_s, \tag{2}$$
$$S_2' = \int [F_{2p} - {}^1/_2\{F_1 + F_1', S_1'\}_p]dt'$$

etc. The curly brackets are the Poisson brackets and the indices p and s denote the periodic and secular part in t' respectively.

the parameter t' is introduced by means of the equations

$$\frac{d(L', G', H')}{dt'} = \frac{\partial F_0'}{\partial(l', g', h')},\quad \frac{d(l', g', h')}{dt'} = -\frac{\partial F_0'}{\partial(L', G', H')}$$

The following Poisson bracket holds by virtue of these equations:

$$\{F_0, S_k\} = -ds_k/dt'$$

and this gives the relation connecting t' and l'

$$dt' = -(\partial F_0/\partial L)^{-1}dl' \tag{3}$$

Applying algorithm (2) to the initial Hamiltonian F, taking (3) into account, we obtain (the prime accompanying the variables are omitted for convenience)

$$F_0' = \frac{G^2 - L^2}{2A} - \frac{L^2}{2C},\quad F_1 = \frac{A-B}{4AB}(G^2 - L^2) \tag{4}$$
$$S_1' = -\frac{C(A-B)}{8BL(A-C)}(G^2 - L^2)\sin 2l - \frac{PACX_c}{LG^2(A-C)}(L\sqrt{G^2 - H^2}\cos l \cos g - G\sqrt{G^2 - H^2}\sin l \sin g + H\sqrt{G^2 - L^2}\cos l)$$

$$F_2' = \left[\frac{C(A-B)^2(G^2 - L^2)(G^2 + 3L^2)}{64AB^2L^2(A-C)} + \frac{kH^2}{4G^4}\sqrt{\frac{G^2 - H^2}{G^2 - L^2}} - \frac{kH^2}{2G^4} + \frac{k}{4L^2}\right] - \frac{k(G^2 - H^2)(G^2 - L^2)}{4L^2G^4}\cos^2 g - \frac{3kH}{4LG^4}\sqrt{(G^2 - H^2)(G^2 - L^2)}\cos g - \frac{kH(G^2 - H^2)}{4LG^4}\cos g,\quad k = \frac{P^2ACX_c^2}{A-C}$$

The resulting Hamiltonian F' does not contain the variable l'. To eliminate the remaining angle variable g' we carry out another canonical transformation

$$L', G', H', l', g', h' \to L'', G'', H'', l'', g'', h''$$

with help of the generating function $S''(L'', G'', H'', l'', g'', h'') = S_1'' + S_2'' + \ldots$. As before, we assume that the Hamiltonian of the new equations of motion has the form $F'' = F_0'' + F_1'' + \ldots$.

To find the components of this Hamiltonian and the generating function, we use the formulas

$$F_0'' = F_0',\ F_1'' = F_1',\ F_2'' = F_{2s}',\ S_1'' = \int F_{2p}'dt'' \tag{5}$$

etc. We introduce the parameter t'' by means of the equations

$$\frac{d(L'', G'', H'')}{dt''} = \frac{\partial F_1''}{\partial(l'', g'', h'')},\quad \frac{d(l'', g'', h'')}{dt''} = -\frac{\partial F_1''}{\partial(L'', G'', H'')}$$

Then the relation connecting t" with g" will be

$$dt'' = -(\partial F_1''/\partial G'')^{-1} dg''$$

Changing in expressions (5) from integration with respect to t" to integration with respect to g", we obtain (omitting the double primes accompanting the variables)

$$F_2'' = [\ldots] - \frac{k(G^2 - H^2)(G^2 - L^2)}{8L^2G^4}$$
$$S_1'' = \frac{k_1(G^2 - H^2)(G^2 - L^2)}{8L^2G^5}\sin 2g + \frac{k_1H(G^2 - H^2)}{2LG^5}\sin g -$$
$$\frac{3k_1H}{2LG^5}\sqrt{(G^2 - H^2)(G^2 - L^2)}; \quad k_1 = \frac{kAB}{A - C}$$

The expressions for F_0'', F_1'' are identical with those given in (4), and the symbol [...] denotes the expression within the square brackets for F_2' in (4).

The Hamiltonian F" obtained contains no angle variables. The equations of motion yield directly $L'' = \text{const}$, $G'' = \text{const}$, $H'' = \text{const}$, while the angle variables will be linear functions of time

$$l'' = l^{\cdot\prime\prime} t + l_0, \quad g'' = g^{\cdot\prime\prime} t + g_0, \quad h'' = h^{\cdot\prime\prime} t + h_0$$

where l_0, g_0, h_0 are the initial values of the corresponding varaibles. From the equations of motion we obtain (double primes accompanying the variables are omitted)

$$l^{\cdot\prime\prime} = -\frac{L(A-C)}{AC} + \frac{L(A-B)}{2AB} + \frac{C(A-B)^2(G^4 + 3L^4)}{32AB^2L^3(A-C)} -$$
$$\frac{k(G^2 + H^2)}{4G^2L^3} - \frac{kLH^2}{4G^4(G^2 - L^2)}\sqrt{\frac{G^2 - H^2}{G^2 - L^2}} + O(\varepsilon^3)$$
$$g^{\cdot\prime\prime} = -\frac{G(A+B)}{2AB} - \frac{CG(A-B)^2(G^2 + L^2)}{16AB^2L^2(A-C)} - \frac{k(G^2 - 10H^2)}{4G^5} - \frac{kH^2}{4L^2G^3} +$$
$$\frac{kH^2[G^2(4G^2 - 3L^2) - H^2(5G^2 - 4L^2)]}{G^5(G^2 - L^2)\sqrt{(G^2 - H^2)(G^2 - L^2)}} + O(\varepsilon^3)$$
$$h^{\cdot\prime\prime} = \frac{5kH}{4G^4} - \frac{kH}{4L^2G^2} - \frac{kH(2G^2 - 3H^2)}{4G^4\sqrt{(G^2 - H^2)(G^2 - L^2)}} + O(\varepsilon^3)$$

The last expression contains only the terms of the second order of smallness in the small parameter.

The initial variables and any function of the initial variables can be found using the formula

$$f(L, G, H, l, g, h) = f(L'', G'', H'', l'', g'', h'') + \{f, S' + S''\} + \tfrac{1}{2}\{f, \{S', S''\}\} + \tfrac{1}{2}\{\{f, S' - S''\}, S' - S''\} + O(\varepsilon^3) \tag{6}$$

Thus the theory in qeustion implies that $H = H^0 = \text{const}$. If we write the terms up to and including the first-order terms in the small parameter, the expressions for the variables *L*, *G* and *h* will be (the double primes accompanying the variables are omitted)

$$L = L - \frac{C(A-B)(G^2 - L^2)}{4BL(A-C)}\cos 2l - \varkappa[L\sqrt{G^2 - H^2}\sin l\cos g +$$
$$G\sqrt{G^2 - H^2}\cos l\sin g + H\sqrt{G^2 - L^2}\sin l] + O(\varepsilon^2)$$
$$G = G - \varkappa[L\sqrt{G^2 - H^2}\cos l\sin g + G\sqrt{G^2 - H^2}\sin l\cos g] +$$
$$\frac{k_1(G^2 - H^2)(G^2 - L^2)}{4L^2G^5}\cos 2g + \frac{k_1H(G^2 - H^2)}{2LG^5}\cos g -$$
$$\frac{3k_1H}{2LG^5}\sqrt{(G^2 - H^2)(G^2 - L^2)}\cos g + O(\varepsilon^2)$$
$$h = h_0 + \varkappa\left[\frac{GH}{\sqrt{G^2 - H^2}}\sin l\sin g + \sqrt{G^2 - L^2}\cos l -\right.$$
$$\left.\frac{LH}{\sqrt{G^2 - H^2}}\cos l\cos g\right] - \frac{k_1H(G^2 - L^2)}{4L^2G^5}\sin 2g +$$
$$\frac{k_1H(G^2 - 3H^2)}{2LG^5}\sin g - \frac{3k_1(G^2 - 2H^2)}{2LG^5}\sqrt{\frac{G^2 - L^2}{G^2 - H^2}}\sin g + O(\varepsilon^2)$$
$$\varkappa = \frac{PACX_c}{LG^2(A-C)}$$

Hence, we obtain the values of these variables in the form of trigonometric functions whose constant coefficients depend on L'', G'', H''. The expressions for the angle variables *l* and *g* contain, in addition to the analogous terms, a secular part. It should be noted that when the theory is constructed including the second-order terms the secular part also appears in the expression for *h*.

Since the Poisson brackets are invariant, formula (6) enables us to determine, fairly simply, in the form of series of the same type, e.g. the variables $p, q, r, \gamma, \gamma', \gamma''$ of the Euler-Poisson problem, provided that we use their expressions in terms of the Deprit variables /4/ and the values obtained for the generating functions S' and S". The latter depend only on the new variables. This significantly facilitates the computations and makes them suitable for computer use.

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ON THE STUDY OF RANDOM OSCILLATIONS IN NON-AUTONOMOUS MECHANICAL SYSTEMS USING THE FOKKER-PLANCK-KOLMOGOROV EQUATIONS*

NGUEN DONG AN

A method of integrating the Fokker-Plank-Kolmogorov equations (FPKE) used in the theory of random oscillations /1—4/ is proposed. The Duffing equation is first studied as an example. The method is then used, together with the method of averaging, to study random oscillations of non-autonomous mechanical systems with one degree of freedom when the eigenfrequency varies in a random manner. The Van-der-Pol equation is considered for the case of a randomly varying eigenfrequency and periodic parametric excitation. When the function sought is replaced, the FPKE transform into another equation whose trivial solutions have the corresponding particular solutions of the FPKE. The condition of integrability of the FPKE is obtained as the direct consequence of the change in question.

1. Consider a mechanical system with one degree of freedom, whose motion is described by the following stochastic equation:

$$x^{\cdot\cdot} - \omega^2 x = f(x, x^{\cdot}) - \sigma\xi^{\cdot}(t) \tag{1.1}$$

$$f(x, x^{\cdot}) = \sum_{s=1}^{m} \alpha_s \sum_{i,\, j=0}^{i+j=s} \gamma_{ij} x^i x^{\cdot j}; \quad \alpha_s, \gamma_{ij} = \mathrm{const} \tag{1.2}$$

where $\xi^{\cdot}(t)$ is a random, white noise-type action of unit intensity. Using the substitution

$$x = a\cos\psi, \; x^{\cdot} = -a\omega\sin\psi \tag{1.3}$$

and the Itc formula, we reduce Eq.(1.1) to the form /4/

$$da = \left[-\frac{1}{\omega} f(a\cos\psi, -a\omega\sin\psi)\sin\psi + \frac{\sigma^2}{2a\omega^2}\cos^2\psi\right] dt - \tag{1.4}$$
$$\frac{\sigma}{\omega}\sin\psi\, d\xi(t)$$
$$d\psi = \left[\omega - \frac{1}{a\omega} f(a\cos\psi, -a\omega\sin\psi)\cos\psi - \frac{\sigma^2}{a^2\omega^2}\sin\psi\cos\psi\right] dt -$$
$$\frac{\sigma}{a\omega}\cos\psi\, d\xi(t)$$

Let us write the FPKE corresponding to system (1.4) for the stationary probability density of the amplitude and phase $W(a, \psi)$

$$\frac{\partial}{\partial a}[B_1(a,\psi)W] + \frac{\partial}{\partial\psi}[B_2(a,\psi)W] = \frac{1}{2}\left\{\frac{\partial^2}{\partial a^2}[B_{11}(a,\psi)W] + \right. \tag{1.5}$$
$$\left. 2\frac{\partial^2}{\partial a\,\partial\psi}[B_{12}(a,\psi)W] + \frac{\partial^2}{\partial\psi^2}[B_{22}(a,\psi)W]\right\}$$

Taking into account the expression for $f(x, x^{\cdot})$ (1.2), we obtain

$$B_1(a,\psi) = -\frac{\sin\psi}{\omega} f(a\cos\psi, -a\omega\sin\varphi) + \frac{\sigma^2\cos^2\psi}{2\omega^2 a} = \frac{\sigma^2\cos^2\psi}{2\omega^2}a^{-1} + \sum_{s=1}^{m} A_s(\psi)a^s \tag{1.6}$$

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